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On ratio asymptotics for general polynomials[☆]

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Abstract

In this paper some characterizations of the ratio asymptotics for general polynomials are given. These results are extensions and improvements of the ratio asymptotics for orthogonal polynomials and are applicable to the ratio asymptotics for polynomials with disturbed nodes.

1. Introduction and main results

Given two triangular matrices of nodes ($n \geqslant 2$):

$$1 \geqslant x_{1,n} > x_{2,n} > \cdots > x_{n,n} \geqslant -1 \tag{1.1}$$

and

$$1 \geqslant y_{1,n-1} > y_{2,n-1} > \cdots > y_{n-1,n-1} \geqslant -1, \tag{1.2}$$

put

$$\omega_n(x) = \prod_{k=1}^{n} (x - x_{k,n}), \quad \Omega_{n-1}(x) = \prod_{k=1}^{n-1} (x - y_{k,n-1}).$$

The main aim of this paper is to give conditions such that the ratio asymptotics

$$\lim_{n\to\infty} \frac{\omega_n(z)}{\Omega_{n-1}(z)} = \frac{1}{2}\phi(z), \quad \phi(z) = z + \sqrt{z^2 - 1} \tag{1.3}$$

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holds for every $z \in \mathbb{C} \setminus \langle -1, 1 \rangle$, where the branch of the square root is chosen so that $|z + \sqrt{z^2 - 1}| > 1$, $z \in \mathbb{C} \setminus [-1, 1]$, and the interval $\langle -1, 1 \rangle$ stands for one of the four intervals $[-1, 1]$, $(-1, 1]$, $[-1, 1)$, and $(-1, 1)$.

To state our results we need some notations. Let us denote by $L[-1, 1]$ the set of complex valued and Riemann integrable functions on $[-1, 1]$ and by $C^1[-1, 1]$ the set of continuously differential complex valued functions on $[-1, 1]$. Write $f_1(x) = \ln(1 - x)$ and $f_2(x) = \ln(1 + x)$. Then the first main result of this paper is as follows.

Theorem 1.1. *Let assumptions* (1.1) *and* (1.2) *prevail and let*

$$1 \geqslant x_{1,n} \geqslant y_{1,n-1} \geqslant x_{2,n} \geqslant y_{2,n-1} \geqslant \cdots \geqslant x_{n-1,n} \geqslant y_{n-1,n-1} \geqslant x_{n,n} \geqslant -1. \tag{1.4}$$

Then the following statements are equivalent:

(a) *relation* (1.3) *holds for every* $z \in \mathbb{C} \setminus [-1, 1]$;

(b) *the relation*

$$\lim_{n \to \infty} \sum_{k=1}^{n} f(x_{k,n}) \frac{\Omega_{n-1}(x_{k,n})}{\omega_n'(x_{k,n})} = \frac{2}{\pi} \int_{-1}^{1} f(x) \sqrt{1 - x^2}\, dx \tag{1.5}$$

holds for every $f \in L[-1, 1]$;

(c) *the relation*

$$\lim_{n \to \infty} \left[\sum_{k=1}^{n} f(x_{k,n}) - \sum_{k=1}^{n-1} f(y_{k,n-1}) \right] = \frac{1}{\pi} \int_{-1}^{1} \frac{f(x)}{\sqrt{1 - x^2}}\, dx \tag{1.6}$$

holds for every $f \in C^1[-1, 1]$.

We also give the other three results concerning the ratio asymptotics (1.3) including the endpoints of the interval $[-1, 1]$.

Theorem 1.2. *Let assumptions* (1.1) *and* (1.2) *prevail and let*

$$1 > x_{1,n} \geqslant y_{1,n-1} \geqslant x_{2,n} \geqslant y_{2,n-1} \geqslant \cdots \geqslant x_{n-1,n} \geqslant y_{n-1,n-1} \geqslant x_{n,n} > -1. \tag{1.7}$$

Then the following statements are equivalent:

(a) *relation* (1.3) *holds for every* $z \in \mathbb{C} \setminus (-1, 1)$;

(b) *the relation*

$$\lim_{n \to \infty} \sum_{k=1}^{n} f(x_{k,n}) \frac{\Omega_{n-1}(x_{k,n})}{\omega_n'(x_{k,n})(1 - x_{k,n}^2)} = \frac{2}{\pi} \int_{-1}^{1} \frac{f(x)}{\sqrt{1 - x^2}}\, dx \tag{1.8}$$

holds for every $f \in L[-1, 1]$;

(c) *relation* (1.6) *holds for every* $f \in C^1[-1, 1] \cup \{f_1, f_2\}$.

Theorem 1.3. *Let assumptions* (1.1) *and* (1.2) *prevail and let*

$$1 > x_{1,n} \geqslant y_{1,n-1} \geqslant x_{2,n} \geqslant y_{2,n-1} \geqslant \cdots \geqslant x_{n-1,n} \geqslant y_{n-1,n-1} \geqslant x_{n,n} \geqslant -1. \tag{1.9}$$

Then the following statements are equivalent:

(a) *relation* (1.3) *holds for every* $z \in \mathbb{C} \backslash [-1, 1)$;
(b) *the relation*

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_{k,n}) \frac{\Omega_{n-1}(x_{k,n})}{\omega'_n(x_{k,n})(1 - x_{k,n})} = \frac{2}{\pi} \int_{-1}^{1} f(x) \sqrt{\frac{1+x}{1-x}}\, dx \tag{1.10}$$

holds for every $f \in L[-1, 1]$;
(c) *relation* (1.6) *holds for every* $f \in C^1[-1, 1] \cup \{f_1\}$.

Theorem 1.4. *Let assumptions* (1.1) *and* (1.2) *prevail and let*

$$1 \geqslant x_{1,n} \geqslant y_{1,n-1} \geqslant x_{2,n} \geqslant y_{2,n-1} \geqslant \cdots \geqslant x_{n-1,n} \geqslant y_{n-1,n-1} \geqslant x_{n,n} > -1. \tag{1.11}$$

Then the following statements are equivalent:

(a) *relation* (1.3) *holds for every* $z \in \mathbb{C} \backslash (-1, 1]$;
(b) *the relation*

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_{k,n}) \frac{\Omega_{n-1}(x_{k,n})}{\omega'_n(x_{k,n})(1 + x_{k,n})} = \frac{2}{\pi} \int_{-1}^{1} f(x) \sqrt{\frac{1-x}{1+x}}\, dx \tag{1.12}$$

holds for every $f \in L[-1, 1]$;
(c) *relation* (1.6) *holds for every* $f \in C^1[-1, 1] \cup \{f_2\}$.

Our investigation of the ratio asymptotics (1.3) for general polynomials is motivated by the ratio asymptotics for orthogonal polynomials. Let $\alpha(x)$ be a nondecreasing function on $[-1, 1]$ with infinitely many points of increase such that all moments of $d\alpha(x)$ are finite and $\{P_n(\alpha; x)\}$,

$$P_n(\alpha; x) = \gamma_n(\alpha)x^n + \cdots, \quad \gamma_n(\alpha) > 0,$$

the orthonormal polynomials with respect to $d\alpha$. We call $d\alpha$ a measure. $\lambda_{k,n}(\alpha)$'s stand for the Christoffel numbers with respect to $d\alpha$ [2, p. 4]. Then we have [2, Theorems 3.2.3 and 3.2.4, pp. 17–19; Theorems 4.1.12 and 4.1.13, pp. 32–34]; [3, Theorem 10]; [8, Theorem 1], in which

$$\omega_n(z) = \Omega_n(z) = \frac{1}{\gamma_n(\alpha)} P_n(\alpha; z).$$

Theorem A. *Let* $d\alpha$ *be a measure supported in* $[-1, 1]$ *and* $x_{k,n}$*'s the zeros of* $P_n(\alpha; x)$. *Then the following statements are equivalent*:

(a) *we have*

$$\lim_{n\to\infty} \frac{\gamma_{n-1}(\alpha)}{\gamma_n(\alpha)} = \frac{1}{2}, \quad \lim_{n\to\infty} \int_{-1}^{1} x P_n(\alpha; x)^2\, d\alpha(x) = 0. \tag{1.13}$$

(b) *relation* (1.3) *holds for every* $z \in \mathbb{C} \backslash (-1, 1)$;

(c) *the relation*

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_{k,n})\lambda_{k,n}(\alpha)P_{n-1}(\alpha; x_{k,n})^2 = \frac{2}{\pi}\int_{-1}^{1} f(x)\sqrt{1-x^2}\,dx \tag{1.14}$$

holds for every $f \in L[-1, 1]$;

(d) *the relation*

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_{k,n})\lambda_{k,n}(\alpha)\frac{P_{n-1}(\alpha; x_{k,n})^2}{1-x_{k,n}^2} = \frac{2}{\pi}\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}\,dx \tag{1.15}$$

holds for every $f \in L[-1, 1]$;

(e) *the relation*

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_{k,n})\lambda_{k,n}(\alpha)\frac{P_{n-1}(\alpha; x_{k,n})^2}{1-x_{k,n}} = \frac{2}{\pi}\int_{-1}^{1} f(x)\sqrt{\frac{1+x}{1-x}}\,dx \tag{1.16}$$

holds for every $f \in L[-1, 1]$;

(f) *the relation*

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_{k,n})\lambda_{k,n}(\alpha)\frac{P_{n-1}(\alpha; x_{k,n})^2}{1+x_{k,n}} = \frac{2}{\pi}\int_{-1}^{1} f(x)\sqrt{\frac{1-x}{1+x}}\,dx \tag{1.17}$$

holds for every $f \in L[-1, 1]$.

Moreover, Statement (a) *implies that relation* (1.6) *holds for every* $f \in C^1[-1, 1]$.

We point out that there is a close relationship between Theorems 1.1–1.4 and Theorem A. In fact, using a well-known formula [1, p. 6]

$$\ell_{k,n}(\alpha; x) = \frac{\gamma_{n-1}(\alpha)}{\gamma_n(\alpha)}\lambda_{k,n}(\alpha)P_{n-1}(\alpha; x_{k,n}) = \frac{P_n(\alpha; x)}{P_n'(\alpha; x_{k,n})}$$

with

$$\ell_{k,n}(\alpha; x) = \frac{P_n(\alpha; x)}{P_n'(\alpha; x_{k,n})(x - x_{k,n})}$$

and substituting $x = x_{k,n}$, we obtain

$$\frac{\gamma_{n-1}(\alpha)}{\gamma_n(\alpha)}\lambda_{k,n}(\alpha)P_{n-1}(\alpha; x_{k,n}) = \frac{1}{P_n'(\alpha; x_{k,n})}.$$

Hence

$$\lambda_{k,n}(\alpha)P_{n-1}(\alpha; x_{k,n})^2 = \frac{\gamma_n(\alpha)P_{n-1}(\alpha; x_{k,n})}{\gamma_{n-1}(\alpha)P_n'(\alpha; x_{k,n})} = \frac{\omega_{n-1}(\alpha; x_{k,n})}{\omega_n'(\alpha; x_{k,n})}$$

and then relations (1.14), (1.15), (1.16), and (1.17) are equivalent to (1.5), (1.8), (1.10), and (1.12), respectively. Thus, Theorems 1.1–1.4 for the ratio asymptotics of general polynomials extend and improve Theorem A for the ratio asymptotics of orthogonal polynomials. The extension of nth root and power asymptotics for orthogonal polynomials to general polynomials can be found in [6,7], respectively.

The second motivation of investigation of the ratio asymptotics for general polynomials is related to study of the asymptotics for polynomials with disturbed nodes. We restrict ourselves to the result for polynomials with disturbed nodes corresponding to Theorem 1.1 in details, the other results will be left to the reader.

Given any two triangular matrices of nodes ($n \geqslant 2$):

$$1 \geqslant \xi_{1,n} > \xi_{2,n} > \cdots > \xi_{n,n} \geqslant -1 \tag{1.18}$$

and

$$1 \geqslant \eta_{1,n-1} > \eta_{2,n-1} > \cdots > \eta_{n-1,n-1} \geqslant -1, \tag{1.19}$$

put

$$\pi_n(x) = \prod_{k=1}^{n} (x - \xi_{k,n}), \quad \Pi_{n-1}(x) = \prod_{k=1}^{n-1} (x - \eta_{k,n-1}).$$

The last main result of this paper is the following.

Theorem 1.5. *Let assumptions* (1.1), (1.2), (1.4), (1.18), *and* (1.19) *prevail and let*

$$1 \geqslant \xi_{1,n} \geqslant \eta_{1,n-1} \geqslant \xi_{2,n} \geqslant \eta_{2,n-1} \geqslant \cdots \geqslant \xi_{n-1,n} \geqslant \eta_{n-1,n-1} \geqslant \xi_{n,n} \geqslant -1. \tag{1.20}$$

If one of the Statements (a)–(c) *of Theorem* 1.1 *is true and if*

$$\lim_{n \to \infty} \left[\sum_{k=1}^{n} |\xi_{k,n} - x_{k,n}| + \sum_{k=1}^{n-1} |\eta_{k,n-1} - y_{k,n-1}| \right] = 0, \tag{1.21}$$

then the following statements hold:

(a) *the relation*

$$\lim_{n \to \infty} \frac{\pi_n(z)}{\Pi_{n-1}(z)} = \frac{1}{2} \phi(z) \tag{1.22}$$

holds for every $z \in \mathbb{C} \backslash [-1, 1]$;

(b) *the relation*

$$\lim_{n \to \infty} \sum_{k=1}^{n} f(\xi_{k,n}) \frac{\Pi_{n-1}(\xi_{k,n})}{\pi_n'(\xi_{k,n})} = \frac{2}{\pi} \int_{-1}^{1} f(x) \sqrt{1 - x^2}\, dx \tag{1.23}$$

holds for every $f \in L[-1, 1]$;

(c) *the relation*

$$\lim_{n\to\infty}\left[\sum_{k=1}^{n} f(\xi_{k,n}) - \sum_{k=1}^{n-1} f(\eta_{k,n-1})\right] = \frac{1}{\pi}\int_{-1}^{1}\frac{f(x)}{\sqrt{1-x^2}}\,dx \tag{1.24}$$

holds for every $f\in C^1[-1,1]$.

The paper is organized as follows. In the next section we state some auxiliary lemmas. In Section 3 we give the proofs of the theorems. In the last section some remarks are given.

2. Auxiliary lemmas

Lemma A (Shi [5, Lemma 2]). *Let assumption* (1.1) *prevail. If* $P(x) = a_0x^{n-1} + a_1x^{n-2} + \cdots + a_{n-1}$, *then*

$$\sum_{k=1}^{n}\frac{P(x_{k,n})}{\omega_n'(x_{k,n})} = a_0. \tag{2.1}$$

Lemma B (Nevai [2, p. 62]). *We have*

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\sqrt{1-x^2}}{z-x}\,dx = \phi(z)^{-1}, \quad z\in\mathbb{C}\backslash(-1,1). \tag{2.2}$$

Lemma C (Saff and Totik [4, Example 3.5, p. 45]). *We have*

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\ln(z-x)}{\sqrt{1-x^2}}\,dx = \ln\frac{\phi(z)}{2}, \quad z\in\mathbb{C}\backslash(-1,1). \tag{2.3}$$

Lemma 2.1. *Let assumptions* (1.1) *and* (1.2) *prevail. If relation* (1.3) *is true for every* $z\in\mathbb{C}\backslash[-1,1]$ *and if*

$$\sup_n \sum_{k=1}^{n}\left|\frac{\Omega_{n-1}(x_{k,n})}{\omega_n'(x_{k,n})}\right| = C_0 < +\infty, \tag{2.4}$$

then relation (1.5) *holds for every* $f\in L[-1,1]$.

Conversely, if relation (1.5) *is true for every* $f\in L[-1,1]$, *then relation* (1.3) *holds for every* $z\in\mathbb{C}\backslash[-1,1]$.

Proof. To prove the first conclusion by Banach–Steinhaus theorem it suffices to show that relation (1.5) holds for every polynomial, or equivalently, for every monomial x^m, $m = 0, 1, \ldots$. To this end using the expansion [1, Formula 1.112-3, p. 21]

$$\phi(z)^{-1} = z - \sqrt{z^2-1} = z[1-(1-z^{-2})^{1/2}] = \sum_{i=0}^{\infty}\frac{(2i-1)!!}{(2i+2)!!}z^{-2i-1}$$

and the formulas [1, Formula 3.621-3, p. 369]

$$\frac{1}{\pi}\int_{-1}^{1} x^{2i}\sqrt{1-x^2}\,dx = \frac{1}{\pi}\int_{0}^{\pi}\cos^{2i}\theta(1-\cos^2\theta)\,d\theta = \frac{(2i-1)!!}{(2i+2)!!},$$

$$\frac{1}{\pi}\int_{-1}^{1} x^{2i+1}\sqrt{1-x^2}\,dx = 0,$$

we obtain the identity

$$\phi(z)^{-1} = \frac{1}{\pi}\sum_{i=0}^{\infty} z^{-i-1}\int_{-1}^{1} x^i\sqrt{1-x^2}\,dx. \tag{2.5}$$

Further, we need the expansion

$$\frac{1}{z-x} = \sum_{i=0}^{\infty} z^{-i-1}x^i, \quad z\in\mathbb{C}\backslash[-1,1]$$

and the Lagrange interpolation formula

$$\Omega_{n-1}(z) = \sum_{k=1}^{n}\frac{\Omega_{n-1}(x_{k,n})\omega_n(z)}{\omega_n'(x_{k,n})(z-x_{k,n})}$$

or equivalently,

$$\frac{\Omega_{n-1}(z)}{\omega_n(z)} = \sum_{k=1}^{n}\frac{\Omega_{n-1}(x_{k,n})}{\omega_n'(x_{k,n})(z-x_{k,n})}. \tag{2.6}$$

By (1.3) we have

$$\begin{aligned} 2\phi(z)^{-1} &= \lim_{n\to\infty}\frac{\Omega_{n-1}(z)}{\omega_n(z)} = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{\Omega_{n-1}(x_{k,n})}{\omega_n'(x_{k,n})(z-x_{k,n})} \\ &= \lim_{n\to\infty}\sum_{k=1}^{n}\frac{\Omega_{n-1}(x_{k,n})}{\omega_n'(x_{k,n})}\sum_{i=0}^{\infty} z^{-i-1}x_{k,n}^i, \end{aligned}$$

which, coupled with (2.5), yields

$$\lim_{n\to\infty}\sum_{k=1}^{n}\sum_{i=0}^{\infty} z^{-i-1}\frac{x_{k,n}^i\Omega_{n-1}(x_{k,n})}{\omega_n'(x_{k,n})} = \frac{2}{\pi}\sum_{i=0}^{\infty} z^{-i-1}\int_{-1}^{1} x^i\sqrt{1-x^2}\,dx. \tag{2.7}$$

Clearly, by (2.1) we have

$$\sum_{k=1}^{n}\frac{\Omega_{n-1}(x_{k,n})}{\omega_n'(x_{k,n})} = 1 = \frac{2}{\pi}\int_{-1}^{1}\sqrt{1-x^2}\,dx,$$

which shows that relation (1.5) holds for the function $f = 1$. Now suppose, as an induction hypothesis, that relation (1.5) holds for every $f(x) = x^i$, $i\leqslant m$. Then it follows from the hypothesis and (2.7) that

$$\lim_{n\to\infty}\sum_{k=1}^{n}\sum_{i=m+1}^{\infty} z^{-i-1}\frac{x_{k,n}^i\Omega_{n-1}(x_{k,n})}{\omega_n'(x_{k,n})} = \frac{2}{\pi}\sum_{i=m+1}^{\infty} z^{-i-1}\int_{-1}^{1} x^i\sqrt{1-x^2}\,dx$$

and hence, multiplying the factor z^{m+2} on both the sides,

$$\lim_{n\to\infty}\sum_{k=1}^{n}\sum_{i=m+1}^{\infty} z^{m+1-i}\frac{x_{k,n}^{i}\Omega_{n-1}(x_{k,n})}{\omega_n'(x_{k,n})}=\frac{2}{\pi}\sum_{i=m+1}^{\infty} z^{m+1-i}\int_{-1}^{1}x^{i}\sqrt{1-x^2}\,dx. \tag{2.8}$$

According to (2.4) we get the estimations

$$\left|\sum_{k=1}^{n}\sum_{i=m+2}^{\infty} z^{m+1-i}\frac{x_{k,n}^{i}\Omega_{n-1}(x_{k,n})}{\omega_n'(x_{k,n})}\right|=\left|\sum_{i=m+2}^{\infty} z^{m+1-i}\sum_{k=1}^{n}\frac{x_{k,n}^{i}\Omega_{n-1}(x_{k,n})}{\omega_n'(x_{k,n})}\right|$$
$$\leqslant\sum_{i=m+2}^{\infty}|z|^{m+1-i}\sum_{k=1}^{n}\left|\frac{\Omega_{n-1}(x_{k,n})}{\omega_n'(x_{k,n})}\right|\leqslant C_0\sum_{i=m+2}^{\infty}|z|^{m+1-i}=\frac{C_0}{|z|-1}$$

and

$$\left|\frac{2}{\pi}\sum_{i=m+2}^{\infty} z^{m+1-i}\int_{-1}^{1}x^{i}\sqrt{1-x^2}\,dx\right|\leqslant\frac{2}{\pi}\sum_{i=m+2}^{\infty}|z|^{m+1-i}\int_{-1}^{1}\sqrt{1-x^2}\,dx=\frac{1}{|z|-1}.$$

Since $|z|$ may be large enough, from (2.8) we conclude that relation (1.5) holds for $f(x)=x^{m+1}$. By induction this proves that relation (1.5) holds for every monomial.

To prove the second conclusion of the lemma inserting $f(x)=1/(z-x)$, $z\in\mathbb{C}\backslash[-1,1]$, into (1.5) it follows from (2.6) and (2.2) that

$$\lim_{n\to\infty}\frac{\Omega_{n-1}(z)}{\omega_n(z)}=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{\Omega_{n-1}(x_{k,n})}{\omega_n'(x_{k,n})(z-x_{k,n})}=\frac{2}{\pi}\int_{-1}^{1}\frac{\sqrt{1-x^2}}{z-x}\,dx=2\phi(z)^{-1}.$$

Since the set

$$\left\{\frac{\Omega_{n-1}(z)}{\omega_n(z)}:\ n\in\mathbb{N}\right\}$$

is a normal family in $\mathbb{C}\backslash[-1,1]$, the convergence in (1.3) holds uniformly in each compact subset of $\mathbb{C}\backslash[-1,1]$. □

Remark 2.1. [2] Applying Banach–Alaoglu theorem and using Stieltjes–Perron inversion formula we can give a more transparent proof of Lemma 2.1 and derive a more general result: Let $M[-1,1]$ be the space of Borel real measure in $[-1,1]$ and K a compact subset of $[-1,1]$. Let $v\in M[-1,1]$ and $z\in\mathbb{C}\backslash K$. Assume that assumptions (1.1) and (1.2) prevail. If

$$\lim_{n\to\infty}\frac{\Omega_{n-1}(z)}{\omega_n(z)}=\int_{-1}^{1}\frac{dv(x)}{z-x}$$

is true and relation (2.4) is valid, then

$$\lim_{n\to\infty}\sum_{k=1}^{n}f(x_{k,n})\frac{\Omega_{n-1}(x_{k,n})}{\omega_n'(x_{k,n})(z-x_{k,n})}=\int_{-1}^{1}f(x)\,dv(x)$$

holds for all $f\in C(K)$.

[2] This interesting remark is due to one of the referees.

We omit the details.

Lemma 2.2. *Let assumptions* (1.1) *and* (1.2) *prevail. If relation* (1.3) *is true for every* $z \in \mathbb{C} \setminus (-1,1)$ *and if*

$$\sup_n \sum_{k=1}^{n} \left| \frac{\Omega_{n-1}(x_{k,n})}{\omega_n'(x_{k,n})(1-x_{k,n}^2)} \right| < +\infty, \tag{2.9}$$

then relation (1.8) *holds for every* $f \in L[-1,1]$.

Conversely, if relation (1.8) *is true for every* $f \in L[-1,1]$, *then relation* (1.3) *holds for every* $z \in \mathbb{C} \setminus (-1,1)$.

Proof. Let us prove the first conclusion of the lemma. Since relation (2.9) implies (2.4), by Lemma 2.1 relation (1.5) holds for every $f \in L[-1,1]$. Inserting $x = 1$ or $x = -1$ into (2.6) give

$$\frac{\Omega_{n-1}(1)}{\omega_n(1)} = \sum_{k=1}^{n} \frac{\Omega_{n-1}(x_{k,n})}{\omega_n'(x_{k,n})(1-x_{k,n})} \tag{2.10}$$

or

$$\frac{\Omega_{n-1}(-1)}{\omega_n(-1)} = -\sum_{k=1}^{n} \frac{\Omega_{n-1}(x_{k,n})}{\omega_n'(x_{k,n})(1+x_{k,n})}, \tag{2.11}$$

respectively. Hence by (1.3)

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{\Omega_{n-1}(x_{k,n})}{\omega_n'(x_{k,n})(1-x_{k,n})} = 2 \tag{2.12}$$

and

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{\Omega_{n-1}(x_{k,n})}{\omega_n'(x_{k,n})(1+x_{k,n})} = 2. \tag{2.13}$$

It follows from (2.12) and (2.13) that

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{\Omega_{n-1}(x_{k,n})}{\omega_n'(x_{k,n})(1-x_{k,n}^2)} = 2, \tag{2.14}$$

which means that relation (1.8) holds for $f = 1$.

Further, since

$$\sum_{k=1}^{n} \frac{x_{k,n}\Omega_{n-1}(x_{k,n})}{\omega_n'(x_{k,n})(1-x_{k,n}^2)} = \sum_{k=1}^{n} \frac{\Omega_{n-1}(x_{k,n})}{\omega_n'(x_{k,n})(1-x_{k,n}^2)} - \sum_{k=1}^{n} \frac{\Omega_{n-1}(x_{k,n})}{\omega_n'(x_{k,n})(1+x_{k,n})},$$

by means of (2.13) and (2.14) we conclude

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{x_{k,n}\Omega_{n-1}(x_{k,n})}{\omega_n'(x_{k,n})(1-x_{k,n}^2)} = 0,$$

which means that relation (1.8) holds for $f(x) = x$. Now suppose, as an induction hypothesis, that relation (1.8) holds for every $f(x) = x^i$, $i \leqslant m$ $(m \geqslant 1)$. Using the recurrence relation

$$\sum_{k=1}^{n} \frac{x_{k,n}^{m+2}\Omega_{n-1}(x_{k,n})}{\omega_n'(x_{k,n})(1-x_{k,n}^2)} = \sum_{k=1}^{n} \frac{x_{k,n}^{m}\Omega_{n-1}(x_{k,n})}{\omega_n'(x_{k,n})(1-x_{k,n}^2)} - \sum_{k=1}^{n} \frac{x_{k,n}^{m}\Omega_{n-1}(x_{k,n})}{\omega_n'(x_{k,n})} \tag{2.15}$$

and the hypothesis it follows from (1.5) that

$$\begin{aligned}\lim_{n\to\infty} \sum_{k=1}^{n} \frac{x_{k,n}^{m+2}\Omega_{n-1}(x_{k,n})}{\omega_n'(x_{k,n})(1-x_{k,n}^2)} &= \frac{2}{\pi}\int_{-1}^{1} \frac{x^m}{\sqrt{1-x^2}}\,dx - \frac{2}{\pi}\int_{-1}^{1} x^m\sqrt{1-x^2}\,dx \\ &= \frac{2}{\pi}\int_{-1}^{1} \frac{x^{m+2}}{\sqrt{1-x^2}}\,dx,\end{aligned}$$

which means that relation (1.8) holds for $f(x) = x^{m+2}$. By induction this proves that relation (1.8) holds for every monomial and hence for every polynomial. By Banach–Steinhaus theorem it follows from (2.9) that relation (1.8) holds for every $f \in L[-1,1]$.

Let us prove the second conclusion of the lemma. If relation (1.8) holds for every $f \in L[-1,1]$, then relation (1.5) also holds for every $f \in L[-1,1]$. According to Lemma 2.1 relation (1.3) holds for every $z \in \mathbb{C}\backslash[-1,1]$. □

Remark 2.2. [3] By the same way as that in Remark 2.1 we can obtain a similar extension of Lemma 2.2.

By the same arguments as that of Lemma 2.2 we can get the following two lemmas, omitting the details.

Lemma 2.3. *Let assumptions* (1.1) *and* (1.2) *prevail. If relation* (1.3) *is true for every* $z \in \mathbb{C}\backslash[-1,1)$ *and if*

$$\sup_n \sum_{k=1}^{n} \left| \frac{\Omega_{n-1}(x_{k,n})}{\omega_n'(x_{k,n})(1-x_{k,n})} \right| < +\infty,$$

then relation (1.10) *holds for every* $f \in L[-1,1]$.

Conversely, if relation (1.10) *is true for every* $f \in L[-1,1]$, *then relation* (1.3) *holds for every* $z \in \mathbb{C}\backslash[-1,1)$.

Lemma 2.4. *Let assumptions* (1.1) *and* (1.2) *prevail. If relation* (1.3) *is true for every* $z \in \mathbb{C}\backslash(-1,1]$ *and if*

$$\sup_n \sum_{k=1}^{n} \left| \frac{\Omega_{n-1}(x_{k,n})}{\omega_n'(x_{k,n})(1+x_{k,n})} \right| < +\infty,$$

then relation (1.12) *holds for every* $f \in L[-1,1]$.

Conversely, if relation (1.12) *is true for every* $f \in L[-1,1]$, *then relation* (1.3) *holds for every* $z \in \mathbb{C}\backslash(-1,1]$.

[3] This interesting remark is also due to one of the referees.

Lemma 2.5. *Let assumptions* (1.1) *and* (1.2) *prevail. If relation* (1.3) *is true for every* $z \in \mathbb{C} \backslash [-1, 1]$ *and if*

$$\sup_n \sum_{k=1}^{n-1} |x_{k,n} - y_{k,n-1}| = C_1 < +\infty, \tag{2.16}$$

then relation (1.6) *holds for every* $f \in C^1[-1, 1]$.

Conversely, if relation (1.6) *is true for every* $f \in C^1[-1, 1]$, *then relation* (1.3) *holds for every* $z \in \mathbb{C} \backslash [-1, 1]$.

Proof. To prove the first conclusion of the lemma we use (1.3) and (2.3) *to obtain*

$$\lim_{n\to\infty} \left[\sum_{k=1}^{n} \ln(z - x_{k,n}) - \sum_{k=1}^{n-1} \ln(z - y_{k,n-1}) \right] = \ln \frac{\phi(z)}{2} = \frac{1}{\pi} \int_{-1}^{1} \frac{\ln(z - x)}{\sqrt{1 - x^2}} dx.$$

Let $P(x) = \prod_{k=1}^{M} (x - z_k), \; z_k \in \mathbb{C} \backslash [-1, 1]$. Then the above relation yields

$$\lim_{n\to\infty} \left[\sum_{k=1}^{n} \ln P(x_{k,n}) - \sum_{k=1}^{n-1} \ln P(y_{k,n-1}) \right] = \frac{1}{\pi} \int_{-1}^{1} \frac{\ln P(x)}{\sqrt{1 - x^2}} dx. \tag{2.17}$$

Now let us show that relation (1.6) holds for every $f(x) = x^m, \; m = 0, 1, \ldots$. Obviously, relation (1.6) holds for $f = 1$. Assume that $m \geqslant 1$ is fixed. Let N be an odd integer and

$$P_1(x) = \sum_{k=0}^{N} \frac{x^{km}}{k!}, \quad P_2(x) = \sum_{k=N+1}^{\infty} \frac{x^{km}}{k!}.$$

Clearly, $P_1(x) > 0, \; P_2(x) > 0, \; x \in [-1, 1]$, and $e^{x^m} = P_1(x) + P_2(x)$. Given an arbitrary positive number ε, choose an odd integer N so large that

$$\left\| \frac{P_2}{P_1} \right\| \leqslant \varepsilon, \quad \left\| \frac{m}{N! P_1} \right\| \leqslant \varepsilon,$$

where $\| \cdot \|$ stands for the uniform norm on $[-1, 1]$. For this fixed number N we write

$$g(x) = \ln \left[1 + \frac{P_2(x)}{P_1(x)} \right].$$

It is easy to see that

$$g(x) = \ln \frac{P_1(x) + P_2(x)}{P_1(x)} = \ln \frac{e^{x^m}}{P_1(x)} = x^m - \ln P_1(x) \tag{2.18}$$

and

$$\|g\| \leqslant \left\| \frac{P_2}{P_1} \right\| \leqslant \varepsilon. \tag{2.19}$$

Meanwhile

$$\begin{aligned} g'(x) &= [x^m - \ln P_1(x)]' = mx^{m-1} - \frac{P_1'(x)}{P_1(x)} \\ &= mx^{m-1} - \frac{\sum_{k=0}^{N} mx^{km-1}/(k-1)!}{P_1(x)} \\ &= mx^{m-1} - \frac{mx^{m-1}[P_1(x) - (x^{mN})/(N!)]}{P_1(x)} \\ &= \frac{mx^{m(N+1)-1}}{N!P_1(x)}. \end{aligned}$$

Hence

$$\|g'\| = \left\| \frac{mx^{m(N+1)-1}}{N!P_1(x)} \right\| \leqslant \left\| \frac{m}{N!P_1} \right\| \leqslant \varepsilon. \tag{2.20}$$

Thus by the mean value theorem for derivative it follows from (2.16), (2.19), and (2.20) that

$$\begin{aligned} &\left| \sum_{k=1}^{n} g(x_{k,n}) - \sum_{k=1}^{n-1} g(y_{k,n-1}) - \frac{1}{\pi} \int_{-1}^{1} \frac{g(x)}{\sqrt{1-x^2}}\,dx \right| \\ &\quad = \left| \sum_{k=1}^{n-1} [g(x_{k,n}) - g(y_{k,n-1})] + g(x_{n,n}) - \frac{1}{\pi} \int_{-1}^{1} \frac{g(x)}{\sqrt{1-x^2}}\,dx \right| \\ &\quad \leqslant \|g'\| \sum_{k=1}^{n-1} |x_{k,n} - y_{k,n-1}| + 2\|g\| \\ &\quad \leqslant (C_1 + 2)\varepsilon. \end{aligned} \tag{2.21}$$

On the other hand, by (2.17) there is a number n_0 such that for $n > n_0$:

$$\left| \sum_{k=1}^{n} \ln P_1(x_{k,n}) - \sum_{k=1}^{n-1} \ln P_1(y_{k,n-1}) - \frac{1}{\pi} \int_{-1}^{1} \frac{\ln P_1(x)}{\sqrt{1-x^2}}\,dx \right| < \varepsilon. \tag{2.22}$$

Thus using (2.18), (2.21), and (2.22) we have that for $n > n_0$:

$$\left| \sum_{k=1}^{n} x_{k,n}^m - \sum_{k=1}^{n-1} y_{k,n-1}^m - \frac{1}{\pi} \int_{-1}^{1} \frac{x^m}{\sqrt{1-x^2}}\,dx \right| < (C_1 + 3)\varepsilon,$$

which proves that relation (1.6) holds for $f(x) = x^m$.

Denote

$$L_n(f) = \sum_{k=1}^{n} f(x_{k,n}) - \sum_{k=1}^{n-1} f(y_{k,n-1}).$$

Then by the mean value theorem for derivative it follows from (2.16) that

$$\begin{aligned}||L_n|| &:= \sup_{\max\{||f||,||f'||\}\leqslant 1} ||L_n(f)|| \\ &= \sup_{\max\{||f||,||f'||\}\leqslant 1} \left|\sum_{k=1}^{n-1} [f(x_{k,n}) - f(y_{k,n-1})] + f(x_{n,n})\right| \\ &= \sup_{\max\{||f||,||f'||\}\leqslant 1} \left|\sum_{k=1}^{n-1} f'(\xi_k)(x_{k,n} - y_{k,n-1}) + f(x_{n,n})\right| \\ &\leqslant \sum_{k=1}^{n-1} |x_{k,n} - y_{k,n-1}| + 1 \\ &\leqslant C_1 + 1.\end{aligned}$$

Then by Banach–Steinhaus theorem relation (1.6) holds for every $f \in C^1[-1,1]$.

Let us prove the second conclusion of the lemma. Inserting $f(x) = \ln(z-x)$ into (1.6) we get

$$\lim_{n\to\infty} \ln\left[\frac{\omega_n(z)}{\Omega_{n-1}(z)}\right] = \frac{1}{\pi}\int_{-1}^{1} \frac{\ln(z-x)}{\sqrt{1-x^2}}\,dx.$$

It remains to apply (2.3). □

3. Proofs of theorems

3.1. Proof of Theorem 1.1. Since relations (1.1), (1.2), and (1.4) imply

$$\frac{\Omega_{n-1}(x_{k,n})}{\omega_n'(x_{k,n})} \geqslant 0, \quad k = 1, 2, \ldots, n,$$

we have by (2.1)

$$\sum_{k=1}^{n} \left|\frac{\Omega_{n-1}(x_{k,n})}{\omega_n'(x_{k,n})}\right| = \sum_{k=1}^{n} \frac{\Omega_{n-1}(x_{k,n})}{\omega_n'(x_{k,n})} = 1. \tag{3.1}$$

Meanwhile relation (1.4) also yields

$$\sum_{k=1}^{n-1} |x_{k,n} - y_{k,n-1}| \leqslant 2. \tag{3.2}$$

Then both (2.4) and (2.16) hold. Hence Theorem 1.1 follows from Lemmas 2.1 and 2.5. □

3.2. Proof of Theorem 1.2. Statement (a) $\Rightarrow$ (b). In the proof of Lemma 2.2 Statement (a) alone implies (2.14). By (1.1), (1.2), (1.7), and (2.14)

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left|\frac{\Omega_{n-1}(x_{k,n})}{\omega_n'(x_{k,n})(1-x_{k,n}^2)}\right| = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{\Omega_{n-1}(x_{k,n})}{\omega_n'(x_{k,n})(1-x_{k,n}^2)} = 2.$$

We conclude that relation (2.9) is true. It remains to apply Lemma 2.2.

(b) $\Rightarrow$ (a). Apply Lemma 2.2.

(a) $\Leftrightarrow$ (c). By Theorem 1.1 it is enough to show that relation (1.3) holds for $x = 1$ (or $x = -1$) if and only if relation (1.6) holds for $f = f_1$ (or $f = f_2$). But this is indeed the case by (2.3). $\square$

3.3. Proof of Theorem 1.3. Use the same argument as that of Theorem 1.2 applying Lemma 2.3 instead of Lemma 2.2. $\square$

3.4. Proof of Theorem 1.4. Use the same argument as that of Theorem 1.2 applying Lemma 2.4 instead of Lemma 2.2. $\square$

3.5. Proof of Theorem 1.5. According to Theorem 1.1 it is sufficient to show that Statement (c) of Theorem 1.1 implies Statement (c) of Theorem 1.5. To this end applying the mean value theorem for derivative we see

$$
\begin{aligned}
&\left|\sum_{k=1}^{n} f(\xi_{k,n}) - \sum_{k=1}^{n-1} f(\eta_{k,n-1}) - \frac{1}{\pi}\int_{-1}^{1}\frac{f(x)}{\sqrt{1-x^2}}dx\right| \\
&\quad = \left|\sum_{k=1}^{n}[f(\xi_{k,n}) - f(x_{k,n})] + \sum_{k=1}^{n-1}[f(y_{k,n-1}) - f(\eta_{k,n-1})]\right. \\
&\qquad \left. + \left[\sum_{k=1}^{n} f(x_{k,n}) - \sum_{k=1}^{n-1} f(y_{k,n-1}) - \frac{1}{\pi}\int_{-1}^{1}\frac{f(x)}{\sqrt{1-x^2}}dx\right]\right| \\
&\quad = \left|\sum_{k=1}^{n} f'(q_{k,n})(\xi_{k,n} - x_{k,n}) + \sum_{k=1}^{n-1} f'(r_{k,n-1})(y_{k,n-1} - \eta_{k,n-1})\right. \\
&\qquad \left. + \left[\sum_{k=1}^{n} f(x_{k,n}) - \sum_{k=1}^{n-1} f(y_{k,n-1}) - \frac{1}{\pi}\int_{-1}^{1}\frac{f(x)}{\sqrt{1-x^2}}dx\right]\right| \\
&\quad \leqslant \|f'\|\left[\sum_{k=1}^{n} |\xi_{k,n} - x_{k,n}| + \sum_{k=1}^{n-1} |y_{k,n-1} - \eta_{k,n-1}|\right] \\
&\qquad + \left|\sum_{k=1}^{n} f(x_{k,n}) - \sum_{k=1}^{n-1} f(y_{k,n-1}) - \frac{1}{\pi}\int_{-1}^{1}\frac{f(x)}{\sqrt{1-x^2}}\right|.
\end{aligned}
$$

Then relation (1.23) follows from (1.21) and (1.6). $\square$

4. Remarks

4.1. Assumption (2.4) is necessary for the first conclusion of Lemma 2.1 in general. In fact, by Banach–Steinhaus theorem it is enough to show that Statement (a) of Theorem 1.1 does not imply (2.4). This is the case from the following example.

Let

$$\begin{cases} \omega_n(x) = 2^{2-n}(x - x_{1n})T_{n-1}(x), \\ \Omega_{n-1}(x) = \omega_{n-1}(x), \end{cases} \tag{4.1}$$

where T_n stands for the nth Chebyshev polynomial of the first kind and

$$x_{1n} = t_{1,n-1} + n^{-n}. \tag{4.2}$$

Then for $z \in \mathbb{C}\backslash[-1, 1]$

$$\lim_{n\to\infty} \frac{\omega_n(z)}{\omega_{n-1}(z)} = \lim_{n\to\infty} \frac{2^{2-n}(z - x_{1,n})T_{n-1}(z)}{2^{3-n}(z - x_{1,n-1})T_{n-2}(z)}$$
$$= \lim_{n\to\infty} \frac{T_{n-1}(z)}{2T_{n-2}(z)} = \frac{1}{2}\phi(z).$$

But

$$\left|\frac{\omega_{n-1}(x_{1,n})}{\omega_n'(x_{1,n})}\right| = \left|\frac{2^{3-n}(x_{1,n} - x_{1,n-1})T_{n-2}(x_{1,n})}{2^{2-n}T_{n-1}(x_{1,n})}\right|$$
$$\geqslant \left|\frac{2(x_{1,n} - x_{1,n-1})(x_{1,n} - x_{2,n-1})}{(x_{1,n} - x_{2,n})(x_{1,n} - x_{n,n})}\right|$$
$$= \left|\frac{2[t_{1,n-1} + n^{-n} - t_{1,n-2} - (n-1)^{-(n-1)}][t_{1,n-1} + n^{-n} - t_{1,n-2}]}{n^{-n}[t_{1,n-1} + n^{-n} - t_{n-1,n-1}]}\right|$$
$$\geqslant n^n[t_{1,n-1} - t_{1,n-2} - (n-1)^{-(n-1)}]^2$$
$$\geqslant n^n\left[\frac{c}{n^2}\right],$$

where $c > 0$ is a certain constant. Thus relation (2.4) is violated.

4.2. Although Statement (b) of Theorem 1.1 is equivalent to Statement (b) of Theorem 1.2 in the case of orthogonal polynomials, this equivalence does not remain true for general polynomials, even if assumption (1.7) is valid.

For example, let ω_n and Ω_{n-1} be given in (4.1) but here

$$x_{1,n} = 1 - n^{-n} \tag{4.3}$$

instead of (4.2). Clearly assumption (1.7) is valid and hence relation (3.1) holds, which implies (2.4). By the same argument as that in Section 4.1 we conclude that Statement (a) of Theorem 1.1 is true and hence Statement (b) of Theorem 1.1 holds. But

$$\frac{\omega_{n-1}(x_{1,n})}{\omega_n'(x_{1,n})(1 - x_{1,n}^2)} = \frac{2^{3-n}(x_{1,n} - x_{1,n-1})T_{n-2}(x_{1,n})}{2^{2-n}T_{n-1}(x_{1,n})(1 - x_{1,n}^2)}$$
$$\geqslant \frac{x_{1,n} - x_{1,n-1}}{1 - x_{1,n}} = \frac{(n-1)^{-(n-1)} - n^{-n}}{n^{-n}}$$
$$= n\left[\frac{n}{n-1}\right]^{n-1} - 1 \geqslant n - 1,$$

which implies that relation (1.8) is not true for $f = 1$.

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References

[1] I.S. Gradshteyn, I.M. Ryzhik, Table of Integrals, Series, and Products, Academic Press, New York, 1980.
[2] P. Nevai, Orthogonal polynomials, Mem. Amer. Math. Soc., Vol. 213, Amer. Math. Soc., Providence, 1979.
[3] P. Nevai, Distribution of zeros of orthogonal polynomials, Trans. Amer. Math. Soc. 249 (1979) 341–361.
[4] E. Saff, V. Totik, Logarithmic Potentials with External Fields, Springer, Berlin, Heidelberg, 1997.
[5] Y.G. Shi, Some extremal problems concerning polynomials, Acta Sci. Math. (Szeged) 61 (1995) 251–259.
[6] Y.G. Shi, Necessary conditions for mean convergence of Lagrange interpolation on arbitrary system of nodes, Acta Math. Hungar. 72 (1996) 251–260.
[7] Y.G. Shi, Necessary conditions for L_p convergence of Lagrange interpolation on arbitrary system of nodes, J. Approx. Theory 85 (1996) 16–26.
[8] Y.G. Shi, New characterizations of ratio asymptotics for orthogonal polynomials, J. Approx. Theory 115 (2002) 1–8.